

Notes on Quantum Field Theory in Curved Spacetime: Problems Relating to the Concept of Particles and Hamiltonian Formalism

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Abstract

The aim of these notes is to elucidate some aspects of quantum field theory in curved spacetime, especially those relating to the notion of particles. A selection of issues relevant to wave-particle duality is given. The case of a generic curved spacetime is outlined. A Hamiltonian formulation of quantum field theory in curved spacetime is elaborated for a preferred reference frame with a separated space metric (a static spacetime and a reductive synchronous reference frame). Applications: (1) Black hole. (2) The universe; the cosmological redshift is obtained in the context of quantum field theory.

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Introduction

One of the essential features of quantum phenomena is the wave-particle duality. It is explicitly represented both in quantum mechanics (eigenstates of coordinate and of momentum) and in quantum field theory in flat spacetime (annihilation/creation operators and field modes). Furthermore, in both theories, the notion of the Hamiltonian is substantial. However, the situation is different in quantum field theory in curved spacetime. As long as a generic curved spacetime is considered [1-8], the concepts of particles and of the Hamiltonian are inconsistent. The reason is that a generic spacetime has no global structure and, therefore, no preferred field modes and vacuum state. However, it is difficult to abandon such vital notions as particle and the Hamiltonian.

In a generic curved spacetime, a Hamiltonian formulation of quantum field theory is inappropriate [1], so a Lagrangian formulation is adopted [1-8]. The situation is different if there exists a preferred reference frame with a separated space metric: $ds^2 = g_{00}(dx^0)^2 + g_{ij}dx^i dx^j$, where $(-g_{ij})$ is a Riemannian metric. Then a Hamiltonian formulation may be implemented. In [9,10], the scalar quantum field has been constructed in a special case of cosmic spacetime.

There are two general cases with the above metric: a synchronous reference frame, where $g_{00} = 1$, $g_{ij} = g_{ij}(x^0, (x^l))$; static spacetime, where $g_{00} = g_{00}((x^l))$, $g_{ij} = g_{ij}((x^l))$. In this paper, both cases are treated from a uniform point of view.

The spin-0, 1, and 1/2 fields and the Hamiltonian are constructed out of annihilation and creation operators. In the case of free fields, there are no divergences. The construction fits into the customary pattern for flat spacetime.

Applications to the universe and the Schwarzschild black hole are given. For the universe (specifically, in the FLRW model) the cosmological redshift is obtained in the context of quantum field theory.

Some issues concerning interaction are considered.

1 Wave-particle duality

1.1 Wave-particle duality in quantum mechanics

One of the most essential features of quantum phenomena is the wave-particle duality. We quote Bohm [11]:

“...Bohr wanted to make the wave-particle duality the starting point of the physical interpretation.

...Bohr developed his conception of complementarity from the wave-particle duality...: There exist complementary properties—like position and momentum—and the exact measurement of one precludes the possibility of obtaining information of the other. Properties are not actualities; they are only possibilities for the physical system. These developments formed the basis of the so-called Copenhagen interpretation of quantum mechanics.”

The simplest quantum mechanical system is a zero-spin particle. Operators relating to the particle are constructed out of \hat{X} (coordinate) and \hat{P} (momentum), a state in the coordinate representation is the wave function $\psi(x)$. A measurement of the coordinate operator \hat{X} would result in a state that has as its classical image a particle. On the other hand, the system in an eigenstate of the momentum operator \hat{P} behaves like a wave.

1.2 Wave-particle duality in quantum field theory

The simplest quantum field system is a scalar field. Operators relating to the field are constructed out of the operator-valued distribution $\hat{\varphi}(x)$, $x = \{x^\mu : \mu = 0, 1, 2, 3\}$, a (pure) state is represented by a vector of the Hilbert space, $\Psi \in \mathcal{H}$.

We quote Kuhlmann [12]:

“Many of the creators of QFT can be found in one of the two camps regarding the question whether particles or fields should be given priority in understanding QFT. While Dirac, the later Heisenberg, Feynman, and Wheeler opted in favor of particles, Pauli, the early Heisenberg, Tomonaga, and Schwinger put fields first...”

We have

$$\hat{\varphi}(x) = \hat{\varphi}^{(+)}(x) + \hat{\varphi}^{(-)}(x), \quad \hat{\varphi}^{(\mp)} = \hat{\varphi}^{(\pm)\dagger} \quad (1.2.1)$$

in the Heisenberg picture

$$\hat{\varphi}^{(+)} = \sum_m \frac{1}{\sqrt{2\omega_m}} f_m(\vec{x}) e^{-i\omega_m t} \hat{a}_m \quad (1.2.2)$$

where the space mode

$$f_m(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{p}_m \vec{x}} \quad (1.2.3)$$

and the frequency

$$\omega_m = \sqrt{\mu^2 + p_m^2} \quad (1.2.4)$$

The Hamiltonian is

$$\hat{H} = \sum_m \omega_m \hat{a}_m^\dagger \hat{a}_m \quad (1.2.5)$$

The wave aspect is represented by the modes $f_m(\vec{x}) e^{-i\omega_m t}$, the particle aspect by the \hat{a}_m , \hat{a}_m^\dagger —in the sense of integrity, which is manifested in the annihilation and creation of particles. We quote Dirac [13]:

“A fraction of a photon is never observed.”

Locality is represented by the transformation

$$\hat{a}'_{m'} = \gamma_{m'}^m \hat{a}_m, \quad F'^{m'} = F^m (\gamma^{-1})_m^{m'} \quad (1.2.6)$$

$$\hat{\varphi}^{(+)} = \sum_m F^m \hat{a}_m = \sum_{m'} F'^{m'} \hat{a}'_{m'}, \quad F^m = \frac{1}{\sqrt{2\omega_m}} f_m e^{-i\omega_m t} \quad (1.2.7)$$

$$\gamma^\dagger \gamma = \gamma \gamma^\dagger = I \quad (1.2.8)$$

with a suitable γ .

As to the role of annihilation/creation operators in quantum theory, we quote Weinberg [14]:

“... creation and annihilation operators were first encountered in the canonical quantization of the electromagnetic field and other fields... They provided a natural formalism for theories in which massive particles as well as photons can be produced and destroyed...”

However, there is a deeper reason for constructing the Hamiltonian out of creation and annihilation operators, which goes beyond the need to quantize any pre-existing field theory like electrodynamics, and has nothing to do with whether particles can actually be produced or

destroyed. The great advantage of this formalism is that if we express the Hamiltonian as a sum of products of creation and annihilation operators, with suitable non-singular coefficients, then the S -matrix will automatically satisfy a crucial physical requirement, the cluster decomposition principle, ... which says in effect that distant experiments yield uncorrelated results. Indeed, it is for this reason that formalism of creation and annihilation operators is widely used in non-relativistic quantum statistical mechanics, where the number of particles is typically fixed. In relativistic quantum theories, the cluster decomposition principle plays a crucial part in making field theory inevitable.”

And Tung [15]:

“(Connection Between Representations of Lorentz and Poincaré Groups.) The c -number wave functions $u^\alpha(\vec{p}\lambda)e^{ipx}$ in the plane wave expansion...are the coefficient functions which connect the set of operators $\{a(\vec{p}\lambda)\}$, transforming as the irreducible unitary representation (m, s) of the Poincaré group, to the set of field operators $\Psi^\alpha(x)$, transforming as certain finite dimensional non-unitary representation of the Lorentz group.

To pursue this group theoretical interpretation of the “plane wave solution” of the wave equation a little further, note that $u^\alpha(\vec{p}\lambda)e^{ipx}$ carries both the Poincaré indices $(\vec{p}\lambda)$ and the Lorentz indices (x, α) .”

2 Quantum fields in a generic curved spacetime

2.1 The problem of the concept of particles

In the case of quantum fields in curved spacetime, the situation changes dramatically. We quote Kay [7]:

“The main new feature of quantum field theory in curved spacetime (present already for linear field theories) is that, in a general (neither flat, nor stationary) spacetime there will not be any single preferred state but rather a family of preferred states, members of which are best regarded as on an equal footing with one-another. It is this feature which makes the above algebraic framework particularly suitable, indeed essential to a clear formulation of the subject. Conceptually, it is this feature which takes the most getting used to. In particular, one must realize that...the interpretation of a state as having a particular “particle-content” is in general problematic because it can only be relative to a particular choice of “vacuum” state and, depending on the spacetime of interest, there may be one state or several states or, frequently, no states at all which deserve the name “vacuum” and even when there are states which deserve this name, they will often only be defined in some approximate or asymptotic or transient sense or only on some subregion of the spacetime.

Concomitantly, one does not expect global observables such as the “particle number” or the quantum Hamiltonian of flat-spacetime free field theory to generalize to a curved spacetime context, and for this reason local observables play a central role in the theory. The quantized stress-energy tensor is a particularly natural and important such local observable and the theory of this is central to the whole subject.”

And Wald [8]:

“Major issues of principle with regard to the formulation of the theory arise from the lack of Poincaré symmetry, the absence of a preferred vacuum state, and, in general, the absence of asymptotic regions in which particle states can be defined...”

The particle interpretation/description of quantum field theory in flat spacetime has been remarkably successful—to the extent that one might easily get impression from the way the theory is normally described that, at a fundamental level, quantum field theory is really a theory of particles. However, the definition of particles relies on the decomposition of ϕ into annihilation and creation operators. . . This decomposition, in turn, relies heavily on the time translation symmetry of Minkowski spacetime, since the “annihilation part” of ϕ is its positive frequency part with respect to time translations. In a curved spacetime that does not possess a time translation symmetry, it is far from obvious how a notion of “particles” should be defined.”

2.2 Divergences

In the conventional treatment, even in the case of free fields, there are divergences in the energy-momentum tensor operator, and the renormalized operator is only defined up to a finite renormalization ambiguity [7], [8].

3 Preferred reference frame

3.1 Reference frame with separated space metric

We now turn to a Hamiltonian formulation of quantum field theory in curved spacetime. It is based first of all on the selection of a reference frame with a preferred time coordinate, so that general covariance is broken. A relevant spacetime manifold is the product manifold:

$$M^{\text{spacetime}} = T^{\text{time}} \times S^{\text{space}} \quad M \ni p = (t, s) \quad t \in T, s \in S \quad (3.1.1)$$

and metric is of the form

$$g = g^{\text{time}} dt \otimes dt + g^{\text{space}} \quad g^{\text{space}} = -h \quad (3.1.2)$$

or

$$ds^2 = g_{00}(dx^0)^2 + g_{ij}dx^i dx^j \quad g_{ij} = -h_{ij} \quad i, j = 1, 2, 3 \quad (3.1.3)$$

where $h = (h_{ij})$ is a Riemannian metric on S . Thus, a selected reference frame is time-orthogonal [16], or that with a separated space metric.

Note that the choice of spatial coordinates is in general immaterial.

3.2 Static spacetime

The first case when metric is of the standard form (3.1.2) is that of a static spacetime, where

$$g^{\text{time}} = g^{\text{time}}(s) \quad h = h(s) \quad s \in S \quad (3.2.1)$$

or

$$g_{00} = g_{00}((x^l)) \quad h_{ij} = h_{ij}((x^l)) \quad (3.2.2)$$

3.3 Synchronous reference frame

The second case with metric of the standard form (3.1.2) is that of a synchronous reference frame, where

$$g^{\text{time}} = 1 \quad h = h_t(s) \quad (3.3.1)$$

or

$$g_{00} = 1 \quad h_{ij} = h_{ij}((x^\mu)) \quad \mu = 0, 1, 2, 3 \quad (3.3.2)$$

3.4 Preferred reference frame

The selection of a preferred reference frame amounts to the choice of a time coordinate. In the case of a static spacetime, the choice is unique. But in the case of a synchronous reference frame, the choice is by no means uniquely defined. In that case, the uniqueness is achieved by the condition that the time t be that of simultaneous quantum state reduction, i.e., quantum jumps over all the space S . This condition defines a reductive reference frame.

Thus, a preferred reference frame is either related to a static spacetime or is a reductive reference frame.

3.5 Energy-momentum tensor and Hamiltonian

In semiclassical gravity, the energy-momentum tensor is defined as

$$T = (\Psi, \hat{T}\Psi) \quad T_\mu^\nu = (\Psi, \hat{T}_\mu^\nu \Psi) \quad (3.5.1)$$

where \hat{T} is the energy-momentum tensor operator and Ψ is a state vector.

The Hamiltonian

$$\hat{H}_t = \int_S \eta \hat{T}_0^0 \quad (3.5.2)$$

where

$$\int_S \eta = \int_S \eta^{\text{space}} := \int_S \sqrt{|h|} d^3x \quad |h| = \det(h_{ij}) \quad (3.5.3)$$

The next problem in the Hamiltonian formulation of quantum field theory is this: Being based on the equivalence principle, to construct the operator \hat{T} for free fields in such a way that the Hamiltonian be of the form

$$\hat{H}_t = \sum_m \omega_m(t) \hat{a}_m^\dagger \hat{a}_m \quad (3.5.4)$$

4 Scalar field

4.1 Energy-momentum tensor and Hamiltonian

For a real scalar field φ , the energy-momentum tensor is of the form

$$T_{\mu\nu} = \varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\sigma\lambda} \varphi_{,\sigma} \varphi_{,\lambda} + \frac{1}{2} g_{\mu\nu} M^2 \varphi^2 \quad (4.1.1)$$

so that for the metric (3.1.3)

$$T_0^0 = \frac{1}{2} g^{00} \varphi_{,0} \varphi_{,0} + \frac{1}{2} h^{jl} \varphi_{,j} \varphi_{,l} + \frac{1}{2} M^2 \varphi^2 \quad (4.1.2)$$

The Hamiltonian

$$H = \int \eta T_0^0 = \frac{1}{2} \int \eta [g^{00} \varphi_{,0} \varphi_{,0} - \varphi \Delta_h \varphi + M^2 \varphi^2] \quad (4.1.3)$$

where Δ_h is the Laplacian on S :

$$\Delta_h \varphi = \frac{1}{\sqrt{|h|}} [\sqrt{|h|} g^{jl} \varphi_{,j}]_{,l} \quad (4.1.4)$$

4.2 Space modes and field operator expansion

Introduce space modes f as solutions to the equation

$$\Delta_h f = -k^2 f \quad f = f(s, t) \quad k^2 = k^2(t) \quad (4.2.1)$$

with the conditions

$$\int \eta f_m^* f_{m'} = \delta_{mm'} \quad (4.2.2)$$

$$(k_m^2 - k_{m'}^2) \int \eta f_m f_{m'} = 0 \quad (4.2.3)$$

For the field operator (in fact, operator-valued distribution), we put in the Schrödinger picture

$$\hat{\varphi}(s, t) = \sum_m \frac{1}{\sqrt{2\omega_m(t)}} [f_m(s, t) \hat{a}_m + f_m^*(s, t) \hat{a}_m^\dagger] \quad (4.2.4)$$

4.3 Inertial time derivation

Now we switch from time derivatives $(\cdots)_{,0}$ to inertial time derivatives $(\cdots)_{:0}$:

$$(\cdots)_{,0} \xrightarrow{\text{switch}} (\cdots)_{:0} \quad (4.3.1)$$

The inertial time derivation is an implementation of the equivalence principle by means of an imitation of the derivation in inertial reference frames in flat spacetime.

For the scalar field (2.2.4) in the Heisenberg picture, the inertial time derivation is defined by

$$\hat{\varphi}_{:0} = \frac{1}{\sqrt{2}} \sum_m \left[\left(\frac{1}{\sqrt{\omega_m}} f_m \hat{a}_m \right)_{:0} + \left(\frac{1}{\sqrt{\omega_m}} f_m^* \hat{a}_m^\dagger \right)_{:0} \right] \quad (4.3.2)$$

$$\left(\frac{1}{\sqrt{\omega_m}} f_m \hat{a}_m \right)_{:0} = -i\sqrt{g_{00}}\omega_m \left(\frac{1}{\sqrt{\omega_m}} f_m \hat{a}_m \right) \quad (4.3.3)$$

$$\left(\frac{1}{\sqrt{\omega_m}} f_m^* \hat{a}_m^\dagger \right)_{:0} = i\sqrt{g_{00}}\omega_m \left(\frac{1}{\sqrt{\omega_m}} f_m^* \hat{a}_m^\dagger \right) \quad (4.3.4)$$

so that

$$\hat{\varphi}_{:0} = i\sqrt{g_{00}} \sum_m \frac{1}{\sqrt{2\omega_m}} [-\omega_m f_m \hat{a}_m + \omega_m f_m^* \hat{a}_m^\dagger] \quad (4.3.5)$$

4.4 The Hamiltonian

We obtain from (4.1.3), (4.2.4), (4.2.1), (4.3.5)

$$\begin{aligned}\hat{H} &= \frac{1}{2} \int \eta [g^{00} \hat{\varphi}_{:0} \hat{\varphi}_{:0} - \hat{\varphi} \Delta_h \hat{\varphi} + M^2 \varphi^2] \\ &= \frac{1}{4} \sum_{mm'} \frac{1}{\sqrt{\omega_m \omega_{m'}}} \int \eta [(-\omega_m \omega_{m'} + k_{m'}^2 + M^2)(f_m f_{m'} \hat{a}_m \hat{a}_{m'} + f_m^* f_{m'}^* \hat{a}_m^\dagger \hat{a}_{m'}^\dagger) \\ &\quad + (\omega_m \omega_{m'} + k_{m'}^2 + M^2)(f_m f_{m'}^* \hat{a}_m \hat{a}_{m'}^\dagger + f_m^* f_{m'} \hat{a}_m^\dagger \hat{a}_{m'})] \quad (4.4.1)\end{aligned}$$

Put

$$\omega_m = \sqrt{M^2 + k_m^2} \quad (4.4.2)$$

then

$$\hat{H} = \frac{1}{2} \sum_m \omega_m (\hat{a}_m \hat{a}_m^\dagger + \hat{a}_m^\dagger \hat{a}_m) \quad (4.4.3)$$

Normal ordering produces in the Schrödinger picture

$$\hat{H}_S(t) = \sum_m \omega_m \hat{a}_m^\dagger \hat{a}_m \quad \hat{a}_m = \hat{a}_{mS} \quad (4.4.4)$$

In the Heisenberg picture,

$$\hat{a}_{mH}(t) = e^{-i\beta_m(t)} \hat{a}_m \quad \hat{a}_{mH}^\dagger(t) = e^{i\beta_m(t)} \hat{a}_m^\dagger \quad \beta(t) = \int_0^t \omega_m(t) dt \quad (4.4.5)$$

and

$$\hat{H}_H(t) = \hat{H}_S(t) \quad (4.4.6)$$

4.5 Charged scalar field

The energy-momentum tensor operator is

$$\hat{T}_{\mu\nu} =: \hat{\varphi}_{,\mu}^\dagger \hat{\varphi}_{,\nu} + \hat{\varphi}_{,\nu}^\dagger \hat{\varphi}_{,\mu} - g_{\mu\nu} g^{\sigma\lambda} \hat{\varphi}_{,\sigma}^\dagger \hat{\varphi}_{,\lambda} + g_{\mu\nu} M^2 \hat{\varphi}^\dagger \hat{\varphi} : \quad (4.5.1)$$

The Hamiltonian

$$\hat{H} = \int \eta : [g^{00} \hat{\varphi}_{:0}^\dagger \hat{\varphi}_{:0} - \hat{\varphi}^\dagger \Delta_h \hat{\varphi} + M^2 \hat{\varphi}^\dagger \hat{\varphi}] : \quad (4.5.2)$$

The field operator

$$\hat{\varphi} = \sum_m \frac{1}{\sqrt{2\omega_m}} [f_m \hat{a}_{(+m)} + f_m^* \hat{a}_{(-m)}^*] \quad (4.5.3)$$

$$\hat{\varphi}^\dagger = \sum_m \frac{1}{\sqrt{2\omega_m}} [f_m^* \hat{a}_{(+m)}^\dagger + f_m \hat{a}_{(-m)}] \quad (4.5.4)$$

We obtain

$$\hat{H}_H(t) = \hat{H}_S(t) = \sum_m \omega_m(t) [\hat{a}_{(-m)}^\dagger \hat{a}_{(-m)} + \hat{a}_{(+m)}^\dagger \hat{a}_{(+m)}] \quad (4.5.5)$$

5 Vector field

5.1 Energy-momentum tensor and Hamiltonian

For a massive vector field A , the energy-momentum tensor is of the form

$$T_{\mu\nu} = \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} - F_{\mu}^{\lambda}F_{\nu\lambda} + M^2(A_{\mu}A_{\nu} - \frac{1}{2}g_{\mu\nu}A_{\lambda}A^{\lambda}) \quad (5.1.1)$$

where

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad (5.1.2)$$

so that for the metric (3.1.3)

$$T_0^0 = \frac{1}{2}[A_{j,l}F^{lj} - M^2A_jA^j] - \frac{1}{2}[A_{0,l}F^{l0} - M^2A_0A^0] - \frac{1}{2}A_{j,0}F^{0j} \quad (5.1.3)$$

The Hamiltonian

$$H = \int \eta T_0^0 \quad (5.1.4)$$

We have

$$\int \eta A_{j,l}F^{lj} = - \int \eta A_j \frac{1}{\sqrt{|h|}}(\sqrt{|h|}F^{lj})_{,l} \quad (5.1.5)$$

and

$$\int \eta A_{0,l}F^{l0} = - \int \eta A_0 \frac{1}{\sqrt{|h|}}(\sqrt{|h|}F^{l0})_{,l} \quad (5.1.6)$$

With the standard metric (3.1.3), $(1/\sqrt{|h|})(\sqrt{|h|}F^{lj})_{,l}$ is a 3-vector and $(1/\sqrt{|h|})(\sqrt{|h|}F^{l0})_{,l}$ is a scalar, so that $(1/\sqrt{|h|})(\sqrt{|h|}F^{l\mu})_{,l}$ makes sense.

5.2 Field operator expansion

Put

$$\hat{A} = \sum_m \frac{1}{\sqrt{2\omega_m}} \sum_{n=1}^3 [f_m e_{mn} \hat{a}_{mn} + f_m^* e_{mn}^* \hat{a}_{mn}^{\dagger}] \quad (5.2.1)$$

where the space modes f_m are defined as in Subsection 4.2, and

$$e_{mn} = e_{mn}(s, t) \quad n = 1, 2, 3 \quad (5.2.2)$$

are the polarization vectors; in components

$$\hat{A}_{\mu} = \sum_m \frac{1}{\sqrt{2\omega_m}} \sum_{n=1}^3 [f_m e_{mn\mu} \hat{a}_{mn} + f_m^* e_{mn\mu}^* \hat{a}_{mn}^{\dagger}] \quad (5.2.3)$$

with orthonormalization conditions

$$e_{mn\mu} e_{mn'}^{*\mu} = -\delta_{nn'} \quad (5.2.4)$$

5.3 Inertial time derivation

Introduce inertial time derivatives:

$$\left(\frac{1}{\sqrt{\omega_m}} f_m e_{mn} \hat{a}_{mn} \right)_{:0} = -i\sqrt{g_{00}}\omega_m \left(\frac{1}{\sqrt{\omega_m}} f_m e_{mn} \hat{a}_{mn} \right) \quad (5.3.1)$$

$$\left(\frac{1}{\sqrt{\omega_m}} f_m^* e_{mn}^* \hat{a}_{mn}^\dagger \right)_{:0} = i\sqrt{g_{00}}\omega_m \left(\frac{1}{\sqrt{\omega_m}} f_m^* e_{mn}^* \hat{a}_{mn}^\dagger \right) \quad (5.3.2)$$

$$\left(\frac{1}{\sqrt{\omega_m}} f_m e_{mn} \hat{a}_{mn} \right)_{:00} = -g_{00}\omega_m^2 \left(\frac{1}{\sqrt{\omega_m}} f_m e_{mn} \hat{a}_{mn} \right) \quad (5.3.3)$$

$$\left(\frac{1}{\sqrt{\omega_m}} f_m^* e_{mn}^* \hat{a}_{mn}^\dagger \right)_{:00} = -g_{00}\omega_m^2 \left(\frac{1}{\sqrt{\omega_m}} f_m^* e_{mn}^* \hat{a}_{mn}^\dagger \right) \quad (5.3.4)$$

In flat spacetime, the equations

$$F^{\nu\mu}_{;\nu} + M^2 A^\mu = 0 \quad (5.3.5)$$

and

$$A^\nu_{;\nu} = 0 \quad (5.3.6)$$

are fulfilled. So we put

$$F^{0\mu}_{:0} + \frac{1}{\sqrt{|h|}} (\sqrt{|h|} F^{l\mu})_{,l} + M^2 A^\mu = 0 \quad (5.3.7)$$

and

$$A^0_{:0} + \frac{1}{\sqrt{|h|}} (\sqrt{|h|} A^l)_{,l} = 0 \quad (5.3.8)$$

Finally, we put

$$[(\cdots)_{,l}]_{:0} =: (\cdots)_{,l:0} = (\cdots)_{:0,l} := [(\cdots)_{:0}]_{,l} \quad (5.3.9)$$

5.4 The Hamiltonian and constraints on the polarization vectors

The Hamiltonian (5.1.4) works out to be

$$H = \frac{1}{2} \int \eta \{ g^{00} [A^\mu (A_\mu)_{:00} - (A^\mu)_{:0} (A_\mu)_{:0}] + g^{00}_{,l} [A^l (A_0)_{:0} - (A^l)_{:0} A_0] \} \quad (5.4.1)$$

Now

$$\begin{aligned} g^{00} : [\hat{A}^\mu (\hat{A}_\mu)_{:00} - (\hat{A}^\mu)_{:0} (\hat{A}_\mu)_{:0}] : &= \frac{1}{2} \sum_{mm'} \left(\frac{\omega_{m'}}{\omega_m} \right)^{1/2} \\ &\times \sum_{nn'} \{ -([(\omega_m + \omega_{m'}) f_m^* f_{m'} e_{mn}^* e_{m'n'\mu} \hat{a}_{mn}^\dagger \hat{a}_{m'n'}] + [\cdots]^\dagger) \\ &+ ([(\omega_m - \omega_{m'}) f_m f_{m'} e_{mn}^{mu} e_{m'n'\mu} \hat{a}_{mn} \hat{a}_{m'n'}] + [\cdots]^\dagger) \} \end{aligned} \quad (5.4.2)$$

and

$$\begin{aligned}
: [\hat{A}^l(\hat{A}_0)_{:0} - (\hat{A}^l)_{:0}\hat{A}_0] : &= \frac{i}{2}\sqrt{g_{00}} \sum_{mm'} \frac{1}{\sqrt{\omega_m\omega_{m'}}} \\
&\times \sum_{nn'} \{ [(\omega_{m'} - \omega_m) f_{m'} f_m e_{m'n'}^l e_{mn0} \hat{a}_{m'n'} \hat{a}_{mn} \\
&+ (\omega_{m'} + \omega_m) f_{m'} f_m^* e_{m'n'}^l e_{mn0}^* \hat{a}_{mn}^\dagger \hat{a}_{m'n'}] - [\dots]^\dagger \} \quad (5.4.3)
\end{aligned}$$

Impose constraints on the (e_{mn}) :

$$e_{mn\mu} e_{mn'}^{*\mu} = -\delta_{nn'} \quad (5.4.4)$$

$$-i\sqrt{g_{00}}\omega_m f_m e_{mn}^0 + \frac{1}{\sqrt{|h|}}(\sqrt{|h|}f_m e_{mn}^l)_{,l} = 0 \quad (5.4.5)$$

$$(\omega_{m'} - \omega_m) \int \eta \sqrt{g_{00}} g^{00}_{,l} f_{m'} f_m e_{m'n'}^l e_{mn0} = 0 \quad (5.4.6)$$

$$\int \eta \sqrt{g_{00}} g^{00}_{,l} f_{m'} f_m^* e_{m'n'}^l e_{mn0}^* = 0 \quad (5.4.7)$$

$$(\omega_m - \omega_{m'}) \int \eta f_m f_{m'} e_{mn}^\mu e_{m'n'\mu} = 0 \quad (5.4.8)$$

$$\int \eta f_m^* f_{m'} e_{mn}^{*\mu} e_{m'n'\mu} = 0 \quad \text{for } m' \neq m \quad (5.4.9)$$

$$\int \eta f_m^* f_m = 1 \quad (5.4.10)$$

Equations (5.4.9), (5.4.10) and (5.4.4) imply

$$\int \eta f_m f_{m'} e_{mn}^\mu e_{m'n'\mu} = -\delta_{mm'} \delta_{nn'} \quad (5.4.11)$$

Then the Hamiltonian

$$\hat{H}_H(t) = \hat{H}_S(t) = \sum_m \omega_m(t) \sum_n \hat{a}_{mn}^\dagger \hat{a}_{mn} \quad (5.4.12)$$

5.5 Massless vector field

We use the radiation gauge:

$$\hat{A}_0 = 0 \quad (5.5.1)$$

Then $n = 1, 2$ and

$$e_{mn}^0 = 0 \quad (5.5.2)$$

6 Dirac field

6.1 Flat spacetime

In flat spacetime, the energy-momentum tensor is of the form

$$T_{\alpha\beta} = \frac{1}{4} \{ [\bar{\psi}(\gamma_{(\alpha} i \partial_{\beta)}) \psi] + [\cdots]^\dagger \} \quad \alpha, \beta = 0, 1, 2, 3 \quad (6.1.1)$$

Here γ^α are Dirac matrices that satisfy the anticommutative relations

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta} \quad \eta = \text{diag}(1, -1, -1, -1) \quad \gamma^{0\dagger} = \gamma^0 \quad \gamma^{a\dagger} = -\gamma^a \quad a = 1, 2, 3 \quad (6.1.2)$$

$$(\alpha \cdots \beta) = \alpha \cdots \beta + \beta \cdots \alpha \quad (6.1.3)$$

$$\bar{\psi} = \psi^\dagger \gamma^0 \quad (6.1.4)$$

The Dirac equation reads

$$i\gamma^\alpha \psi_{,\alpha} - M\psi = 0 \quad (6.1.5)$$

6.2 Curved spacetime

The curved spacetime generalization of the above formulas is given by the following replacements [2]:

$$\gamma^\alpha \rightarrow \gamma^\mu = V_\alpha^\mu \gamma^\alpha \quad \mu = 0, 1, 2, 3 \quad (6.2.1)$$

$$\partial_\alpha \rightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu \quad (6.2.2)$$

where

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (6.2.3)$$

$$\Gamma_\mu = \frac{1}{2} \Sigma^{\alpha\beta} V_\alpha^\nu V_{\beta\nu,\mu} \quad \Sigma^{\alpha\beta} = \frac{1}{4} [\gamma^\alpha, \gamma^\beta] \quad (6.2.4)$$

$$V_\alpha^\mu V_\beta^\nu g_{\mu\nu} = \eta_{\alpha\beta} \quad (6.2.5)$$

$$V_\alpha^\mu = V_{(\alpha)}^\mu \quad \{V_{(\alpha)} : \alpha = 0, 1, 2, 3\} \text{ is a tetrad} \quad (6.2.6)$$

Now

$$T_{\mu\nu} = \frac{1}{4} \{ [\bar{\psi}(\gamma_{(\mu} i \nabla_{\nu)}) \psi] + [\cdots]^\dagger \} \quad (6.2.7)$$

and

$$i\gamma^\mu \nabla_\mu \psi - M\psi = 0 \quad (6.2.8)$$

Write (6.2.7) as

$$T_{\mu\nu} = \psi^\dagger \frac{1}{4} (K_{\mu\nu} + K_{\mu\nu}^\dagger) \psi \quad (6.2.9)$$

where

$$K_{\mu\nu} = \gamma^0 [\gamma_{(\mu} i \nabla_{\nu)}] = K_{\nu\mu} \quad (6.2.10)$$

specifically,

$$K_{00} = 2\gamma^0 \gamma_0 i \nabla_0 \quad (6.2.11)$$

6.3 Standard metric

We now turn to the case of the standard metric (3.1.3). Put

$$V_{(0)}^j = 0 \quad j = 1, 2, 3 \quad (6.3.1)$$

Then

$$\gamma^0 = V_{(0)}^0 \gamma_D^0 \quad \gamma_D^0 := \gamma_{\text{Dirac}}^0 = \gamma^{\alpha=0} \quad [V_{(0)}^0]^2 = g^{00} \quad (6.3.2)$$

The Hamiltonian

$$H = \int \eta T_0^0 \quad (6.3.3)$$

so that we are interested mainly in

$$T_0^0 = g^{00} T_{00} = g^{00} \psi^\dagger \frac{1}{4} (K_{00} + K_{00}^\dagger) \psi \quad (6.3.4)$$

with K_{00} given by (6.2.11).

6.4 Inertial time derivation

Introduce an inertial time derivative:

$$\nabla_0 \xrightarrow{\text{switch}} \nabla_{:0} \quad (6.4.1)$$

Since

$$\nabla_0 = \partial_0 + \Gamma_0 \quad (6.4.2)$$

we consider

$$\partial_0 \xrightarrow{\text{switch}} \partial_{:0} \quad (6.4.3)$$

In the spirit of the inertial time derivation, we take

$$\partial_{:0} g^{00} = g^{00} \partial_{:0} \quad \nabla_{:0} g^{00} = g^{00} \nabla_{:0} \quad (6.4.4)$$

and, accordingly, put

$$T_0^0 = \psi^\dagger \Theta_0^0 \psi \quad (6.4.5)$$

where

$$\Theta_0^0 = \frac{1}{4} [(g^{00} K_{00}) + (g^{00} K_{00})^\dagger] \quad [\Theta_0^0]^\dagger = \Theta_0^0 \quad (6.4.6)$$

$$K_{00} = 2\gamma^0 \gamma_0 i \nabla_{:0} \quad (6.4.7)$$

The next step is as follows. With (6.2.8) in mind, we put

$$i \nabla_{:0} = (\gamma^0)^{-1} [\gamma^l (-i \nabla_l) + M] \quad (6.4.8)$$

so that

$$K_{00} = 2\gamma^0 \gamma_0 (\gamma^0)^{-1} [\gamma^l (-i \nabla_l) + M] \quad (6.4.9)$$

Now,

$$\gamma^0 \gamma_0 (\gamma^0)^{-1} = V_{(0)}^0 g_{00} \gamma_D^0 \quad (6.4.10)$$

and

$$g^{00}K_{00} = 2V_{(0)}^0\gamma_D^0[\gamma^l(-i\nabla_l) + M] \quad (6.4.11)$$

Note that with (6.4.8) the relation (6.4.4) does not hold; it was introduced only to arrive at the expression (6.4.6).

We obtain

$$(g^{00}K_{00})^\dagger = 2\gamma_D^0[\gamma^l(-i\nabla_l) + M]V_{(0)}^0 \quad (6.4.12)$$

so that finally

$$\Theta_0^0 = \frac{1}{2}\gamma_D^0[\{V_{(0)}^0\gamma^l, -i\nabla_l\} + 2V_{(0)}^0M] \quad [V_{(0)}^0]^2 = g^{00} \quad (6.4.13)$$

where $\{\dots, \dots\}$ is an anticommutator.

6.5 Space modes and field operator

Introduce space modes by the equation

$$\Theta_0^0 f_m = E_m f_m \quad f_m = f_m(s, t) \quad E_m = E_m(t) \quad (6.5.1)$$

and orthonormalize them according to

$$\int \eta f_{m'}^\dagger f_m = 2\omega_m \delta_{m'm} \quad (6.5.2)$$

where

$$\omega_m = |E_m| \quad E_m \neq 0 \quad (6.5.3)$$

Put

$$\hat{\psi} = \sum_{m \neq 0} \frac{1}{\sqrt{2\omega_m}} f_m [\theta(E_m)\hat{a}_m + \theta(-E_m)\hat{b}_m^\dagger] \quad (6.5.4)$$

$$\hat{\psi}^\dagger = \sum_{m \neq 0} \frac{1}{\sqrt{2\omega_m}} f_m^\dagger [\theta(E_m)\hat{a}_m^\dagger + \theta(-E_m)\hat{b}_m] \quad (6.5.5)$$

6.6 The energy-momentum tensor operator and the Hamiltonian

The energy-momentum tensor operator is

$$\hat{T}_{\mu\nu} =: \hat{\psi}^\dagger \frac{1}{4} (K_{\mu\nu} + K_{\mu\nu}^\dagger) \hat{\psi} : \quad (6.6.1)$$

where $K_{\mu\nu}$ is given by (6.2.10) with the switch (6.4.1), (6.4.8).

The Hamiltonian

$$\hat{H} = \int \eta : \hat{T}_0^0 := \int \eta : \hat{\psi}^\dagger \Theta_0^0 \hat{\psi} : \quad (6.6.2)$$

with Θ_0^0 given by (6.4.13). We have

$$\Theta_0^0 f_m \theta(\pm E_m) = \theta(\pm E_m) E_m f_m = \theta(\pm E_m) (\pm \omega_m) f_m \quad (6.6.3)$$

so that

$$\hat{H} = \sum_m \omega_m [\theta(E_m) \hat{a}_m^\dagger \hat{a}_m + \theta(-E_m) \hat{b}_m^\dagger \hat{b}_m] \quad (6.6.4)$$

$$E_m = E_m(t) \quad \omega_m = \omega_m(t) \quad \hat{H} = \hat{H}_t \quad (6.6.5)$$

$$\hat{H}_H = \hat{H}_S = \hat{H}_t \quad (6.6.6)$$

6.7 Reductive reference frame

In a reductive reference frame, we have

$$g_{00} = 1 \quad g^{00} = 1 \quad V_{(0)}^0 = 1 \quad (6.7.1)$$

so that

$$\Theta_0^0 = \gamma_D^0 \left[\frac{1}{2} \{ \gamma^l, -i \nabla_l \} + M \right] \quad (6.7.2)$$

and

$$[\Theta_0^0]^\dagger = \left[-\frac{1}{2} \{ \gamma^l, -i \nabla_l \} + M \right] \gamma_D^0 = \Theta_0^0 \quad (6.7.3)$$

Consider

$$\Theta_0^0 \Theta_0^0 = [\Theta_0^0]^\dagger \Theta_0^0 = -\frac{1}{4} \{ \gamma^j, -i \nabla_j \} \{ \gamma^l, -i \nabla_l \} + M^2 \quad (6.7.4)$$

Introduce

$$\mathcal{P} := \frac{1}{2} \{ \gamma^l, -i \nabla_l \} \quad \mathcal{P}^\dagger = -\mathcal{P} \quad (6.7.5)$$

Thus

$$[\Theta_0^0]^2 = \mathcal{P}^\dagger \mathcal{P} + M^2 \quad (6.7.6)$$

We have

$$[\Theta_0^0]^2 f_m = E_m^2 f_m \quad (6.7.7)$$

or

$$\mathcal{P}^\dagger \mathcal{P} f_m = (E_m^2 - M^2) f_m \quad (6.7.8)$$

which implies

$$E_m^2 - M^2 \geq 0 \quad (6.7.9)$$

Thus

$$\mathcal{P}^\dagger \mathcal{P} f_m = k_m^2 f_m \quad (6.7.10)$$

and

$$E_m^2 = M^2 + k_m^2 \quad E_m = \pm \omega_m \quad \omega_m = \sqrt{M^2 + k_m^2} \quad (6.7.11)$$

6.8 Diagonal space metric

Let a space metric be diagonal:

$$g^{\text{space}} = \sum_l g_{ll} dx^l \otimes dx^l = - \sum_l h_{ll} dx^l \otimes dx^l = -h \quad (6.8.1)$$

Then the vectors $\partial/\partial x^j$, $j = 1, 2, 3$, are mutually orthogonal:

$$h \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^{j'}} \right) = \delta_{jj'} h_{jj} \quad (6.8.2)$$

and it is expedient to choose tetrad vectors along them:

$$(V_{(a)} : a = 1, 2, 3) = (V_{(j)} : j = 1, 2, 3) \quad (6.8.3)$$

$$V_{(j)} = V_{(j)}^j \frac{\partial}{\partial x^j} \quad (\text{no } \sum_j) \quad [V_{(j)}^j]^2 = h^{jj} \quad h(V_{(j)}, V_{(j')}) = \delta_{jj'} \quad (6.8.4)$$

Now we have

$$(V_{(\alpha)} : \alpha = 0, 1, 2, 3) = (V_{(\mu)} : \mu = 0, 1, 2, 3) \quad (6.8.5)$$

$$V_{(\mu)} = V_{(\mu)}^\mu \frac{\partial}{\partial x^\mu} \quad (\text{no } \sum_\mu) \quad [V_{(\mu)}^\mu]^2 = |g^{\mu\mu}|^2 \quad g(V_{(\mu)}, V_{(\mu')}) = \eta_{\mu\mu'} \quad (6.8.6)$$

From (6.2.4) follows

$$\Gamma_\mu = 0 \quad (6.8.7)$$

so that

$$\nabla_\mu = \partial_\mu \quad \partial_0 \xrightarrow{\text{switch}} \partial_{,0} \quad (6.8.8)$$

Next,

$$\gamma^l = V_{(l)}^l \gamma_D^l \quad (\text{no } \sum_l) \quad [V_{(l)}^l]^2 = h^{ll} \quad (6.8.9)$$

For a reductive reference frame with a diagonal metric,

$$\Theta_0^0 = \gamma_D^0(\mathcal{P} + M) \quad (6.8.10)$$

$$\mathcal{P} = \pm \frac{1}{2} \sum_l \gamma_D^l \{ \sqrt{h^{ll}}, -i\partial_l \} \quad (6.8.11)$$

7 Massless Weyl field

7.1 Flat spacetime

Let us briefly review the massless Weyl field. In flat spacetime, the energy-momentum tensor is of the form

$$T_{\alpha\beta} = \sum_H^{R,L} T_{H\alpha\beta} \quad (7.1.1)$$

$$T_{H\alpha\beta} = \frac{1}{2}\psi_H^\dagger\{[\sigma_{(\alpha}P_H\mathbf{i}\partial_{\beta)}] + [\cdot\cdot]^\dagger\}\psi_H \quad (7.1.2)$$

Here

$$\text{Hand} =: H = R, L := \text{Right, Left} \quad (7.1.3)$$

σ^a , $a = 1, 2, 3$, are the Pauli matrices, $\sigma^0 = I$,

$$P_L = 1 \quad P_R\mathbf{i}\partial_0 = \mathbf{i}\partial_0 \quad P_R\mathbf{i}\partial_a = -\mathbf{i}\partial_a \quad (7.1.4)$$

The equation

$$\sigma^\alpha(P_H\mathbf{i}\partial_\alpha)\psi_H = 0 \quad (7.1.5)$$

holds.

7.2 Curved spacetime

The energy-momentum tensor

$$T_{\mu\nu} = \sum_H^{R,L} T_{H\mu\nu} \quad (7.2.1)$$

$$T_{H\mu\nu} = \psi_H^\dagger \frac{1}{2} (K_{H\mu\nu} + K_{H\mu\nu}^\dagger) \psi_H \quad (7.2.2)$$

where

$$K_{H\mu\nu} = \sigma_{(\mu} P_H \mathbf{i} \nabla_{\nu)} \quad (7.2.3)$$

Equation (7.1.5) is replaced with

$$\sigma^\mu (P_H \mathbf{i} \nabla_\mu) \psi_H = 0 \quad (7.2.4)$$

7.3 Standard metric and inertial time derivation

The switch from ∇_0 to $\nabla_{:0}$ is

$$P_H \mathbf{i} \nabla_0 \xrightarrow{\text{switch}} P_H \mathbf{i} \nabla_{:0} = (\sigma^0)^{-1} \sigma^l P_H (-\mathbf{i} \nabla_l) \quad (7.3.1)$$

The Hamiltonian

$$H = \sum_H^{R,L} H_H \quad (7.3.2)$$

$$H_H = \int \eta T_{H0}^0 \quad (7.3.3)$$

and

$$T_{H0}^0 = \psi_H^\dagger \Theta_{H0}^0 \psi_H \quad (7.3.4)$$

where

$$\Theta_{H0}^0 = \frac{1}{2} [(g^{00} K_{H00}) + (g^{00} K_{H00})^\dagger] \quad (7.3.5)$$

$$K_{H00} = \sigma_{(0} P_H \mathbf{i} \nabla_{:0)} = 2\sigma_0 (\sigma^0)^{-1} \sigma^l P_H (-\mathbf{i} \nabla_l) \quad (7.3.6)$$

7.4 Space modes, field operators, and Hamiltonian

The equation for space modes is

$$\Theta_{H0}^0 f_{Hm} = E_{Hm} f_{Hm} \quad \omega_{Hm} = |E_{Hm}| \quad (7.4.1)$$

Field operators are

$$\hat{\psi}_H = \sum_m^{E_{Hm} \neq 0} \frac{1}{\sqrt{\omega_{Hm}}} f_{Hm} [\theta(E_{Hm}) \hat{a}_{Hm} + \theta(-E_{Hm}) \hat{b}_{Hm}^\dagger] \quad (7.4.2)$$

where

$$\bar{H} = \begin{cases} L & H = R \\ R & H = L \end{cases} \quad (7.4.3)$$

The Hamiltonian is

$$\hat{H} = \sum_H^{R,L} \hat{H}_H \quad (7.4.4)$$

$$\hat{H}_H = \sum_m \omega_{Hm} (\hat{a}_{Hm}^\dagger \hat{a}_{Hm} + \hat{b}_{Hm}^\dagger \hat{b}_{Hm}) \quad (7.4.5)$$

8 On quantum field state vector

8.1 The Einstein equation in semiclassical gravity

In semiclassical gravity, the Einstein equation reads

$$G - \Lambda g = 8\pi \varkappa(\Psi, \hat{T}\Psi) \quad (8.1.1)$$

where G is the Einstein tensor, Λ is the cosmological constant, \varkappa is the gravitational constant, and Ψ is a state vector.

8.2 Constraints on state vector

We consider a family of quantum fields,

$$\Phi = \{\hat{\varphi}, \hat{A}, \hat{\psi}, \dots\} \quad (8.2.1)$$

for a given metric g . So (8.1.1) may be written as

$$(G - \Lambda g)[g] = 8\pi \varkappa(\Psi, \hat{T}[\Phi[g], g]\Psi) \quad (8.2.2)$$

where $[g]$ means a dependence on metric and its derivatives. This equation imposes constraints on Ψ . Here are two examples.

If g describes a vacuum spacetime, then $\Psi = \Psi_{\text{vac}}$.

If a spacetime is static, then Ψ is stationary: $\hat{H}\Psi = E\Psi$.

9 The universe

9.1 The closed universe

We consider the closed universe. Cosmic space is a three-sphere:

$$S^{\text{space}} = S^{\text{cosmic}} = S^3 = \{x_k : k = 1, 2, 3, 4, \sum_k x_k^2 = 1\} \quad (9.1.1)$$

Introduce the radius of the universe, $R(t)$. We have for the space volume

$$V(t) = \int \eta = \int_{S^3} |h_t| d^3x \quad (9.1.2)$$

Put

$$R := (V/2\pi^2)^{1/3} \quad R = R(t) \quad (9.1.3)$$

and

$$h_t = R^2(t)\varpi_t \quad \int_{S^3} \eta_\varpi = \int_{S^3} \sqrt{|\varpi|} d^3x = 2\pi^2 \quad (9.1.4)$$

Now

$$g = dt \otimes dt - R^2 \varpi_t \quad ds^2 = dt^2 - R^2 \varpi_{ij} dx^i dx^j \quad (9.1.5)$$

Thus, we have a reductive reference frame.

9.2 Scalar and vector fields

The equation for space modes (4.2.1) takes the form

$$\frac{1}{R^2} \Delta_\varpi f = -k^2 f \quad (9.2.1)$$

or

$$\Delta_\varpi f = -\tilde{k}^2 f \quad (9.2.2)$$

$$k^2(t) = \frac{\tilde{k}_t^2}{R^2(t)} \quad (9.2.3)$$

Thus

$$\omega_m(t) = \sqrt{M^2 + \frac{\tilde{k}_{mt}^2}{R^2(t)}} \quad (9.2.4)$$

9.3 Dirac field

We have

$$\Theta_0^0 = \gamma_D^0(\mathcal{P} + M) \quad \mathcal{P} = \frac{1}{2} \{\gamma^l, -i\nabla_l\} \quad (9.3.1)$$

where

$$\gamma^l = V_{(a)}^l \gamma^a \quad \nabla_l = \partial_l + \Gamma_l \quad \Gamma_l = \frac{1}{8} [\gamma^a, \gamma^b] V_{(a)}^j V_{(b)j,l} \quad a, b = 1, 2, 3 \quad (9.3.2)$$

Now,

$$h^{jn}V_{(a)j}V_{(b)n} = h_{jn}V_{(a)}^jV_{(b)}^n = \delta_{ab} \quad (9.3.3)$$

and

$$h_{jn} = R^2\varpi_{jn} \quad h^{jn} = \frac{1}{R^2}\varpi^{jn} \quad (9.3.4)$$

so that

$$V_{(a)}^l = \frac{v_{(a)}^l}{R} \quad V_{(a)l} = Rv_{(a)l} \quad v_{(a)}^l = \varpi^{lj}v_{(a)j} \quad (9.3.5)$$

Thus,

$$\mathcal{P} = \frac{1}{R}Q \quad (9.3.6)$$

$$Q = \frac{1}{2}\{v_{(a)}^l\gamma^a, -i(\partial_l + \frac{1}{8}[\gamma^a\gamma^b]v_{(a)}^jv_{(b)j,l})\} \quad (9.3.7)$$

Now (6.7.10) reads

$$\frac{1}{R^2}Q^\dagger Q f_m = k_m^2 f_m \quad (9.3.8)$$

or

$$Q^\dagger Q f_m = \tilde{k}_m^2 f_m \quad (9.3.9)$$

$$k_m^2(t) = \frac{\tilde{k}_{mt}^2}{R^2(t)} \quad (9.3.10)$$

so that

$$E_m^2(t) = M^2 + \frac{\tilde{k}_{mt}^2}{R^2(t)} \quad E_m = \pm\omega_m \quad \omega_m(t) = \sqrt{M^2 + \frac{\tilde{k}_{mt}^2}{R^2(t)}} \quad (9.3.11)$$

9.4 Massless Weyl field

We obtain

$$K_{H00} = \frac{2}{R}v_{(a)}^l\sigma^a P_H(-i\nabla_l) \quad \sigma^a = \sigma_{\text{Pauli}}^a \quad (9.4.1)$$

$$\Theta_{H0}^0 = \frac{1}{R}\Xi_H \quad (9.4.2)$$

where

$$\Xi_H = \sigma^a\{v_{(a)}^l, P_H(-i\nabla_l)\} \quad (9.4.3)$$

Now (7.4.1) reads

$$\frac{1}{R}\Xi_H f_{Hm} = E_{Hm} f_m \quad (9.4.4)$$

so that

$$E_{Hm} = \pm\omega_{Hm} \quad \omega_{Hm}(t) = \frac{\tilde{k}_{Hmt}}{R(t)} \quad \tilde{k}_{Hm} \geq 0 \quad (9.4.5)$$

9.5 The FLRW universe

Now consider the Friedmann-Lemaître-Robertson-Walker model of the universe. The Robertson-Walker metric is of the form

$$ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right] \quad (9.5.1)$$

It is diagonal with

$$\varpi = \frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (9.5.2)$$

independent of time.

Here the quantity \hat{k}_m is time independent for all fields.

9.6 Cosmological redshift

For all fields, the result

$$\omega_m(t) = \sqrt{M^2 + \frac{\tilde{k}_m^2}{R^2(t)}} \quad (9.6.1)$$

has been obtained. Specifically, for photons

$$\omega_m(t) = \frac{\tilde{k}_m}{R(t)} \quad (9.6.2)$$

which represents the cosmological redshift. What is essential, is that the result has been obtained in the context of quantum field theory.

10 Black hole spacetime

10.1 The Schwarzschild metric

In this section, the Schwarzschild black hole is considered. The Schwarzschild metric is of the form

$$ds^2 = (1 - r_S/r)dt^2 - \frac{1}{1 - r_S/r}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (10.1.1)$$

where $r_S = 2M_S$ is the Schwarzschild radius. Introduce dimensionless quantities:

$$\tilde{r} = \frac{r}{r_S} \quad \tilde{t} = \frac{t}{r_S} \quad d\tilde{s}^2 = \frac{ds^2}{r_S^2} \quad (10.1.2)$$

$$d\tilde{s}^2 = (1 - 1/\tilde{r})d\tilde{t}^2 - \frac{1}{1 - 1/\tilde{r}}d\tilde{r}^2 - \tilde{r}^2(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (10.1.3)$$

This metric is static and diagonal. It has a physical singularity at $\tilde{r} = 0$, which is unavoidable. In addition, it involves a coordinate singularity at $\tilde{r} = 1$, though $0 < \tilde{r} < \infty$. To eliminate the latter, we have to introduce a synchronous reference frame.

10.2 Complete synchronous reference frame

We will use a complete synchronous reference frame [17] which is defined as follows. Metric is

$$ds^2 = d\tau^2 - e^{\lambda(\tau, \varrho)} d\varrho^2 - r^2(\tau, \varrho)(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (10.2.1)$$

where

$$r = \frac{1}{2} r_S \left[\left(\frac{\varrho}{r_S} \right)^2 + 1 \right] (1 + \cos \eta) \quad (10.2.2)$$

$$\tau = \frac{1}{2} r_S \left[\left(\frac{\varrho}{r_S} \right)^2 + 1 \right]^{3/2} (\eta + \sin \eta) \quad (10.2.3)$$

$$e^\lambda = \frac{1}{4} \left[\left(\frac{\varrho}{r_S} \right)^2 + 1 \right] \frac{[2(1 + \cos \eta)^2 + 3(\sin \eta)(\eta + \sin \eta)]^2}{(1 + \cos \eta)^2} \quad (10.2.4)$$

$$-\pi < \eta < \pi \quad -\infty < \varrho < \infty \quad (10.2.5)$$

and it is implicit that (10.2.3) determines

$$\eta = \eta(\tau, \varrho) \quad (10.2.6)$$

Since a change

$$(t, r) \xrightarrow{\text{change}} (\tau, \varrho) \quad (10.2.7)$$

is made, the variable η may be regarded as representing the variable t .

With $-\infty < \varrho < \infty$, the reference frame under consideration involves two black and two white holes. To reduce this to only one black and one white hole, we make a change

$$\varrho \xrightarrow{\text{change}} \xi = \varrho^2 \quad (10.2.8)$$

Introducing dimensionless quantities

$$\tilde{\tau} = \frac{\tau}{r_S} \quad \tilde{\xi} = \tilde{\varrho}^2 = \left(\frac{\varrho}{r_S} \right)^2 \quad (10.2.9)$$

we obtain

$$d\tilde{s}^2 = d\tilde{\tau}^2 - \frac{e^\lambda}{4\tilde{\xi}} d\tilde{\xi}^2 - \tilde{r}^2(d\theta^2 + \sin^2 \theta d\varphi^2) \quad \tilde{r} > 0 \quad \tilde{\xi} > 0 \quad (10.2.10)$$

$$\tilde{r} = \frac{1}{2}(\tilde{\xi} + 1)(1 + \cos \eta) \quad (10.2.11)$$

$$\tilde{\tau} = \frac{1}{2}(\tilde{\xi} + 1)^{3/2}(\eta + \sin \eta) \quad (10.2.12)$$

$$e^\lambda = \frac{1}{4}(\tilde{\xi} + 1) \frac{[2(1 + \cos \eta)^2 + 3(\sin \eta)(\eta + \sin \eta)]^2}{(1 + \cos \eta)^2} \quad (10.2.13)$$

with

$$-\pi < \eta < \pi \quad (10.2.14)$$

In addition to the physical singularity at $\tilde{r} = 0$, the metric (10.2.10) has a coordinate singularity at $\tilde{\xi} = 0$, but the latter is unessential as $\tilde{\xi} = 0$ is a boundary point.

From (10.2.12) and (10.2.14) it follows that

$$\tilde{\xi} > \theta \left(|\tilde{\tau}| - \frac{\pi}{2} \right) \left[\left(\frac{2|\tilde{\tau}|}{\pi} \right)^{2/3} - 1 \right] \quad (10.2.15)$$

10.3 Geometry of spacetime

Let us analyze the geometry of the spacetime with the metric (10.1.3), (10.2.10),

$$(1 - 1/\tilde{r})d\tilde{t}^2 - \frac{1}{1 - 1/\tilde{r}}d\tilde{r}^2 = d\tilde{\tau}^2 - \frac{e^\lambda}{4\tilde{\xi}}d\tilde{\xi}^2 \quad (10.3.1)$$

(1) $\tilde{r} = \text{const}$

(10.3.1) reduces to

$$d\tilde{t} = (1 + 1/\tilde{\xi})^{1/2}d\tilde{\tau} \quad \tilde{r} \neq 1 \quad (10.3.2)$$

with $\tilde{\xi} = \tilde{\xi}(\tilde{\tau}, \tilde{r})$ via (10.2.11), (10.2.12).

Let

$$\eta = \pi - \delta \quad \delta \ll 1 \quad \tilde{r} > 1 \quad (10.3.3)$$

then

$$\tilde{r} = \frac{1}{4}(\tilde{\xi} + 1)\delta^2 \quad \tilde{\tau} = \frac{1}{2}(\tilde{\xi} + 1)^{3/2}(\pi - \delta^3/6) \quad \xi \gg 1 \quad (10.3.4)$$

$$\tilde{\tau} = \frac{\pi}{2}(\tilde{\xi} + 1)^{3/2} - \frac{2}{3}\tilde{r}^{3/2} \quad (10.3.5)$$

and

$$\tilde{t}_2 - \tilde{t}_1 = \tilde{\tau}_2 - \tilde{\tau}_1 + \frac{3}{2} \left(\frac{\pi}{2} \right)^{2/3} (\tilde{\tau}_2^{1/3} - \tilde{\tau}_1^{1/3}) \quad \tilde{\tau} \gg 1 \quad (10.3.6)$$

Let

$$\eta \ll 1 \quad (10.3.7)$$

then

$$\tilde{r} = (\tilde{\xi} + 1)(1 - \eta^2/4) \approx \tilde{\xi} + 1 \quad \tilde{\tau} = (\tilde{\xi} + 1)^{3/2}\eta \quad (10.3.8)$$

and for $\tilde{r} > 1$

$$\tilde{\xi} = \tilde{r} - 1 > 0 \quad \tilde{t}_2 - \tilde{t}_1 = (1 + 1/\tilde{\xi})^{1/2}(\tilde{\tau}_2 - \tilde{\tau}_1) \quad \frac{\tilde{\tau}_2 - \tilde{\tau}_1}{\tilde{r}^{3/2}} \ll 1 \quad (10.3.9)$$

(2) $\tilde{\xi} = \text{const}$

(10.3.1) reduces to

$$d\tilde{t} = \left(\frac{\tilde{r}}{\tilde{r} - 1} \right)^{1/2} \left[1 + \frac{\tilde{r}}{\tilde{r} - 1} \frac{1}{\tilde{\xi} + 1} \frac{\sin^2 \eta}{(1 + \cos \eta)^2} \right]^{1/2} d\tilde{\tau} \quad (10.3.10)$$

Let

$$\eta \ll 1 \quad (10.3.11)$$

then

$$d\tilde{t} = \left(\frac{\tilde{\xi} + 1}{\tilde{\xi}} \right)^{1/2} \left[1 + \frac{1}{4\tilde{\xi}(\tilde{\xi} + 1)^3} \tilde{\tau}^2 \right] d\tilde{\tau} \quad (10.3.12)$$

Let

$$\tilde{r} = 1 + w \quad 0 < w \ll 1 \quad (10.3.13)$$

then

$$\tilde{t} = \tilde{t}_1 + \log \frac{w_1}{w} \quad w = w_1 - \frac{1}{2}(\tilde{\xi} + 1)^{1/2} \left[1 - \left(\frac{2}{\tilde{\xi} + 1} - 1 \right)^2 \right] (\tilde{\tau} - \tilde{\tau}_1) =: w_1 - \frac{w_1}{\tilde{\tau}_0(\tilde{\xi})} (\tilde{\tau} - \tilde{\tau}_1) \quad (10.3.14)$$

so that

$$w = w_1 \left[1 - \frac{\tilde{\tau} - \tilde{\tau}_1}{\tilde{\tau}_0(\tilde{\xi})} \right] \quad \tilde{t} = \tilde{t}_1 + \log \frac{1}{1 - (\tilde{\tau} - \tilde{\tau}_1)/\tilde{\tau}_0(\tilde{\xi})} \quad (10.3.15)$$

$$\tilde{t} \rightarrow \infty \quad \text{for } \tilde{\tau} \rightarrow \tilde{\tau}_1 + \tilde{\tau}_0(\tilde{\xi}) \quad (10.3.16)$$

(3) $\tilde{\tau} = \text{const}$

(10.3.1) reduces to

$$d\tilde{t} = \frac{1}{4} \left(\frac{\tilde{r}}{\tilde{r} - 1} \right)^{1/2} \left[\frac{\tilde{r}}{\tilde{r} - 1} - \frac{\tilde{\xi} + 1}{\tilde{\xi}} \right]^{1/2} \frac{2(1 + \cos \eta)^2 + 3(\sin \eta)(\eta + \sin \eta)}{1 + \cos \eta} d\tilde{\xi} \quad (10.3.17)$$

Let

$$\eta \ll 1 \quad \tilde{r} > 1 \quad (10.3.18)$$

then

$$d\tilde{t} = \frac{\tilde{\tau}}{\tilde{\xi}} \left[\frac{\tilde{\xi} + 1}{4\tilde{\xi}(\tilde{\xi} + 1)^2 - \tilde{\tau}^2} \right]^{1/2} d\tilde{\xi} \quad (10.3.19)$$

10.4 Quantum fields

For scalar and vector fields, we obtain the Laplacian

$$\tilde{\Delta}_{\tilde{\xi}\theta\varphi} = \tilde{\Delta}_{\tilde{\xi}} + \frac{1}{\tilde{r}^2} \Delta_{\theta\varphi} \quad \tilde{\tau} = \text{const} \quad (10.4.1)$$

where

$$\tilde{\Delta}_{\tilde{\xi}} f = \frac{4\sqrt{\tilde{\xi}}}{\tilde{r}^2 e^{\lambda/2}} \frac{\partial}{\partial \tilde{\xi}} \left[\sqrt{\tilde{\xi}} \frac{\tilde{r}^2}{e^{\lambda/2}} \frac{\partial f}{\partial \tilde{\xi}} \right] \quad (10.4.2)$$

Formulas for spin-1/2 fields may be obtained straightforwardly.

A state vector complying to vacuum spacetime is the vacuum one:

$$\Psi = \Psi_{\text{vac}} \quad (10.4.3)$$

There is no ambiguity here since quantum fields are constructed in a preferred reference frame.

11 On interaction

11.1 Divergences

There are two primary sources of difficulties in quantum field theory, specifically in scattering theory, which manifest themselves in divergences:

- (1) Fields as operator-valued distributions rather than functions.
- (2) Perturbation theory, specifically the Dyson expansion, the expansion of the Green functions, and the expansion in path-integral methods.

A field at a fixed x is not an (unbounded) operator. The expression $e^{-i\hat{H}t}$ makes sense only if \hat{H} is an (unbounded) selfadjoint operator. For free fields, this does not give rise to difficulties, but for interacting fields the difficulties are well known.

If \hat{H} is an unbounded operator, the expansion of $e^{-i\hat{H}t}$ in the powers of $(-i\hat{H}t)$ may be incorrect [18].

11.2 Cutoff

The simplest way to transform a field into an operator is to introduce cutoff for space modes:

$$f_m \mapsto f_m \zeta(\omega_m) \quad (11.2.1)$$

There is a possibility to introduce $\zeta(\omega_m)$ in a natural way. Introduce the smooth function

$$\eta(x) = \begin{cases} 1 & \text{for } x \leq 1 \\ 1 - \exp\{e^{-x}/(1-x)\} & \text{for } x > 1 \\ 0 & \text{for } x = \infty \end{cases} \quad (11.2.2)$$

and

$$\zeta(\omega) = \eta\left(\frac{\omega^2}{1/\kappa} + \frac{\kappa\Lambda^2}{\omega^2}\right) \quad (11.2.3)$$

where κ is the gravitational constant ($\kappa = t_P^2$, t_P is the Planck time) and Λ is the cosmological constant. We have

$$\kappa\Lambda^2 \lesssim \omega^2 \lesssim 1/\kappa \quad (11.2.4)$$

i.e., both an ultraviolet and an infrared cutoff. For special relativity, $\kappa \rightarrow 0$, the cutoff vanishes:

$$0 \leq \omega^2 \leq \infty \quad (11.2.5)$$

11.3 Product dynamics

As long as the Hamiltonian is an (unbounded) selfadjoint operator—due to the cutoff—it is possible to use product dynamics [18-20]. It is this:

$$\Psi(t_2) = \hat{U}(t_2, t_1)\Psi(t_1) \quad (11.3.1)$$

$$\hat{U}(t_2, t_1) = T \exp \left\{ -i \int_{t_1}^{t_2} \hat{H}(t) dt \right\}, \quad \hat{H}^\dagger(t) = \hat{H}(t) \quad (11.3.2)$$

$$T \exp \left\{ -i \int_{t_1}^{t_2} \hat{H}(t) dt \right\} = \lim_{N \rightarrow \infty} e^{-i\hat{H}(t_2 - \Delta t/N) \Delta t/N} e^{-i\hat{H}(t_2 - 2\Delta t/N) \Delta t/N} \dots e^{-i\hat{H}(t_1) \Delta t/N} \quad (11.3.3)$$

$\Delta t = t_2 - t_1 > 0$

Now

$$\hat{U}^\dagger(t_2, t_1) = T^\dagger \exp \left\{ i \int_{t_1}^{t_2} \hat{H}(t) dt \right\} = \lim_{N \rightarrow \infty} e^{i\hat{H}(t_1) \Delta t/N} \dots e^{i\hat{H}(t_2 - \Delta t/N) \Delta t/N} \quad (11.3.4)$$

so that

$$\hat{U}^\dagger(t_2, t_1) \hat{U}(t_2, t_1) = \hat{U}(t_2, t_1) \hat{U}^\dagger(t_2, t_1) = I \quad (\text{unitarity}), \quad \hat{U}^\dagger(t_2, t_1) = \hat{U}(t_1, t_2) \quad (11.3.5)$$

This approach is better than the Dyson expansion [18-20] (in the sense of convergence, not computability).

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